Total marks – 80
Attempt Questions 1–5

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 *(14 Marks)* Use a SEPARATE writing booklet **Marks**

(a) For the curve. $y = 4 + 3x - x^3$,

(i) find any stationary points and determine their nature. **4**

(ii) find any points of inflexion. **2**

(iii) find the co-ordinates of the *y*-intercept. **1**

(iv) sketch the curve in the domain $-3 \le x \le 3$ showing all the above features. **2**

(b) The sum of the radii of two circles is 100 *cm*. If one of the circles has a radius of *x cm*

(i) Show that the sum of the areas of the two circles is given by **2**

$$A = 2\pi(x^2 - 100x + 5000).$$

(ii) Find the least possible value for this area. **3**

Question 2 *(14 Marks)* Use a SEPARATE writing booklet

(a) Find the centre and radius of this circle $x^2 + 14x + 14 + y^2 - 2y = 0$. **3**

(b) Express $5x^2 + 2x - 3$ in the form $A(x+1)^2 + B(x+1) + C$. **3**

(c) For what values of *k* does the equation $kx^2 - 4kx - (k-5) = 0$ have real roots ? **3**

Question 2 continues on Page 2.

Question 2 Continued. **Marks**

(d) The vertex of the parabola is $(-7,-2)$ and the focus is $(-3,-2)$. Find

(i) the equation of the directrix. **1**

(ii) the equation of the parabola. **1**

(e) Derive the equation of the locus of the point $P(x, y)$ that moves so that it is equidistant from the point $A(-6,5)$ and the point B $(3,-1)$. **3**

Question 3 *(16 Marks)* Use a SEPARATE writing booklet

(a) Find:

(i) $\displaystyle\int (4x^3 + 7x^2 - 3)\,dx$. **1**

(ii) $\displaystyle\int \frac{6x^3 - 7x}{x^2}\,dx$. **2**

(iii) $\displaystyle\int x\sqrt{x}\,dx$ **2**

(b) Evaluate $\displaystyle\int_0^2 (2x-1)^3\,dx$. **2**

(c) Find the equation of the curve that passes through the point (2, 5), given that the gradient function is $3+2x-x^2$. **3**

(d) (i) Sketch, on the same axes $y=10-x^2$ and $y=x+4$. **2**

(ii) Hence find the area bounded by parabola $y=10-x^2$ and the line $y=x+4$. **4**

Question 4 *(16 Marks)* Use a SEPARATE writing booklet **Marks**

(a) Differentiate

(i) e^{4x+8} **1**

(ii) $\dfrac{e^x}{e^x+1}$ **2**

(b) Evaluate $\displaystyle\int_0^2 e^{3x}\,dx$. **2**

(c) Find $\displaystyle\int 8x^3 e^{x^4}\,dx$. **2**

(d) Find the equation of the tangent to the curve $y = e^{x^2-1}$ at $x = 1$. **3**

(e) Find the exact volume of the solid of revolution formed when the curve $y = e^x - e^{-x}$ is rotated about the x-axis between $x = 0$ and $x = \dfrac{1}{2}$. **3**

(f) Use Simpson's Rule with five function values to find an approximation to $\displaystyle\int_0^4 e^{-x^2}\,dx$, correct to two decimal places. **3**

Question 5 *(20 Marks)* Use a SEPARATE writing booklet

(a) Find $\dfrac{dy}{dx}$ given

(i) $y = \log_e(2x+7)$ **1**

(ii) $y = \log_e(2x+1)(x-5)$ **2**

(iii) $y = x^3 \log_e x$ **2**

(b) Evaluate $\displaystyle\int_2^3 \frac{8x}{2x^2+7}\,dx$. **2**

Question 5 continues on Page 4.

Question 5 Continued. **Marks**

(c) (i) Show $\dfrac{4x+3}{2x+1}=2+\dfrac{1}{2x+1}$. **1**

(ii) Hence find $\displaystyle\int \frac{4x+3}{2x+1}\,dx$ **2**

(d) Simplify $2\log_3 6+\log_3 18-3\log_3 2$. **3**

(e) Solve

(i) $\log_2 64 = x$ **1**

(ii) $3^x = 5$, correct to two decimal places. **2**

(f) (i) Sketch the curve $y=\log_e x$. **1**

(ii) Find the area enclosed between the curve $y=\log_e x$, the x axis and the line $x=2$. Shade this area on your diagram. **3**

END OF THE PAPER

Question 1

a) $y = 4 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

$\frac{dy}{dx^2} = -6x$

For stationary points $\frac{dy}{dx} = 0$

$3 - 3x^2 = 0$

$3(1 - x^2) = 0$

$3(1-x)(1+x) = 0$

$x = \pm 1$

when $x = 1$

$y = 4 + 3x - x^3$

$= 4 + 3(1) - (1)^3$

$= 6$

$\therefore (1,6)$ is a stationary point.

Test the nature

$\frac{d^2y}{dx^2} = -6x$ at $x = 1$

$= -6(1)$

$= -6$

As $\frac{d^2y}{dx^2} < 0$ the curve is concave down

$\therefore (1,6)$ is a local maximum

when $x = -1$

$y = 4 + 3(x) - x^3$

$= 4 + 3(-1) - (-1)^3$

$= 2$

$\therefore (-1,2)$ is a stationary point.

Test the nature

$\frac{d^2y}{dx^2} = -6x$ at $x = -1$

$= -6(-1)$

$= 6$

As $\frac{d^2y}{dx^2} > 0$ the curve is concave up

$\therefore (-1,2)$ is a local minimum.

ii) For points of inflexion $\frac{d^2y}{dx^2} = 0$ and there is a change in concavity

$\frac{d^2y}{dx^2} = -6x$

$-6x = 0$

$x = 0$

$y = 4 + 3x - x^3$ at $x = 0$

$= 4 + 3(0) - 0$

$= 4$

$\therefore (0,4)$ may be a point of inflexion

check concavity

x	0^-	0	0^+
$\frac{d^2y}{dx^2} = -6x$	+		−

$\therefore$ There is a change in concavity

$\therefore (0,4)$ is a point of inflexion

(iii) the y intercept is $(0,4)$

(iv) Endpoints

when $x = -3$, $y = 4 + 3x - x^3$

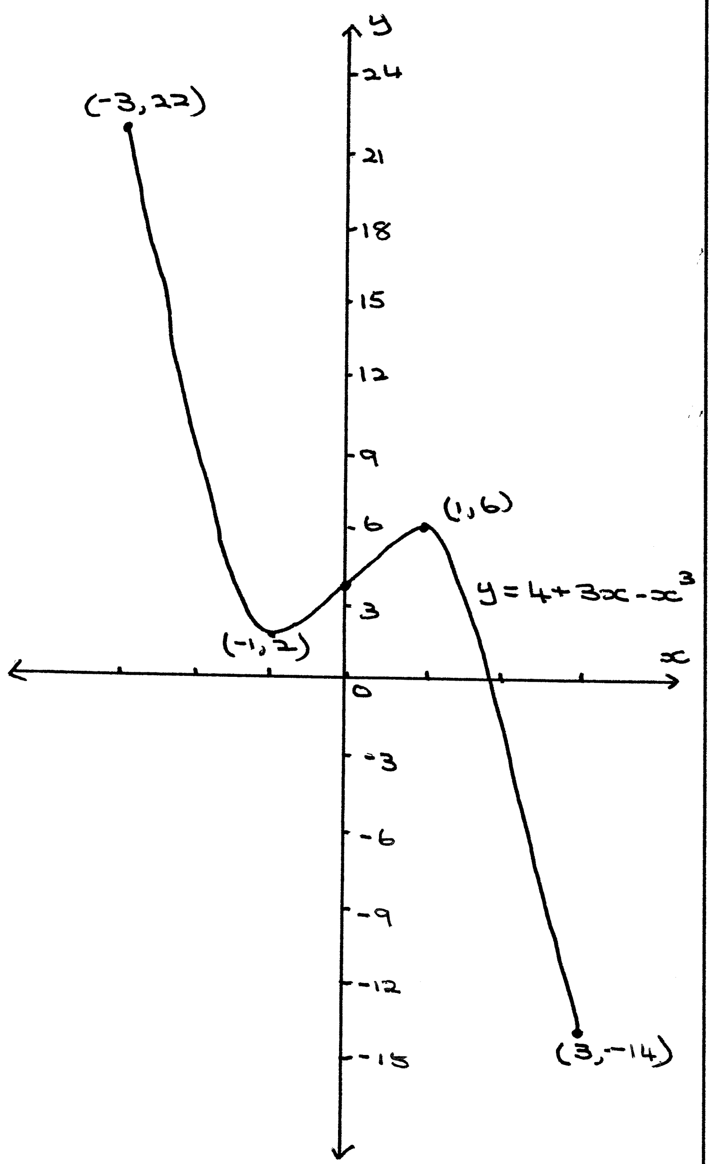
$= 4 + 3(-3) - (-3)^2$

$= 22$

when $x = 3$

$$y = 4 + 3x - x^3$$
$$= 4 + 3(3) - 3^3$$
$$= -14$$

$\therefore$ The endpoints are $(-3, 22)$ and $(3, 14)$



b) radius of 1 circle $= x$

radius of the other circle $= 100 - x$

$$A = \pi r^2 + \pi R^2$$
$$= \pi x^2 + \pi(100 - x)^2$$
$$= \pi x^2 + 10000\pi - 200\pi x + \pi x^2$$
$$= 2\pi x^2 - 200\pi x + 10000\pi$$
$$\therefore A = 2\pi(x^2 - 100x + 5000)$$

ii) $A = 2\pi x^2 - 200\pi x + 10000\pi$

$$\frac{dA}{dx} = 4\pi x - 200\pi$$

$$\frac{d^2A}{dx^2} = 4\pi$$

For stationary points $\frac{dA}{dx} = 0$

$$\therefore 4\pi x - 200\pi = 0$$
$$4\pi x = 200\pi$$
$$x = 50$$

when $x = 50$

$$A = 2\pi x^2 - 200\pi x + 10000\pi$$
$$= 2\pi(50)^2 - 200\pi(50) + 10000\pi$$
$$= 5000\pi$$

As $\frac{d^2y}{dx^2} > 0$, a minimum value occurs

$\therefore$ The least possible area is $5000\pi \text{ cm}^2$

Question 2

a) $x^2 + 14x + 14 + y^2 - 2y = 0$

$$x^2 + 14x + \left(\frac{14}{2}\right)^2 + y^2 - 2y + \left(\frac{-2}{2}\right)^2 = -14 + \left(\frac{14}{2}\right)^2 + \left(\frac{-2}{2}\right)^2$$

$$(x+7)^2 + (y-1)^2 = -14 + 49 + 1$$
$$(x+7) + (y-1)^2 = 36$$
$$(x+7) + (y-1)^2 = 6^2$$

$\therefore$ centre is $(-7, 1)$

radius is 6

b) $5x^2 + 2x - 3$

$$= A(x+1)^2 + B(x+1) + C$$
$$= Ax^2 + 2Ax + A + Bx + B + C$$
$$= Ax^2 + (2A+B)x + (A+B+C)$$

$\therefore A = 5$

$2A + B = 2$
$2(5) + B = 2$
$B = -8$

$A + B + C = -3$
$5 - 8 + C = -3$
$C = 0$

$$\therefore 5x^2 + 2x - 3 \equiv 5(x+1)^2 - 8(x+1)$$

c) $kx^2 - 4kx - (k-5) = 0$

$\Delta = b^2 - 4ac$

$= (-4k)^2 - 4(k) \times -(k-5)$

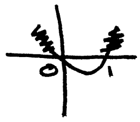
$= 16k^2 + 4k^2 - 20k$

$= 20k^2 - 20k$

For real roots $\Delta \geq 0$

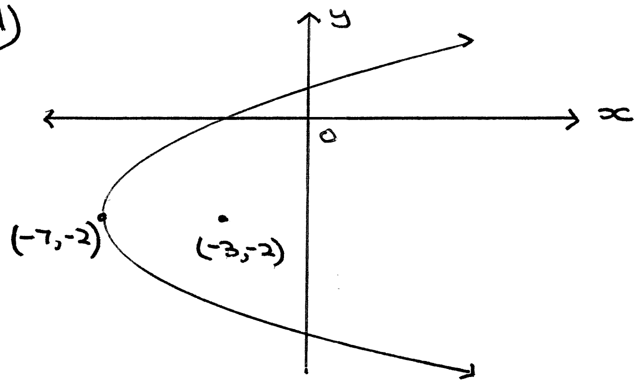
$\therefore 20k^2 - 20k \geq 0$

$20k(k-1) \geq 0$



$\therefore k \leq 0, \ k \geq 1$

d)



(i) $a = 4$

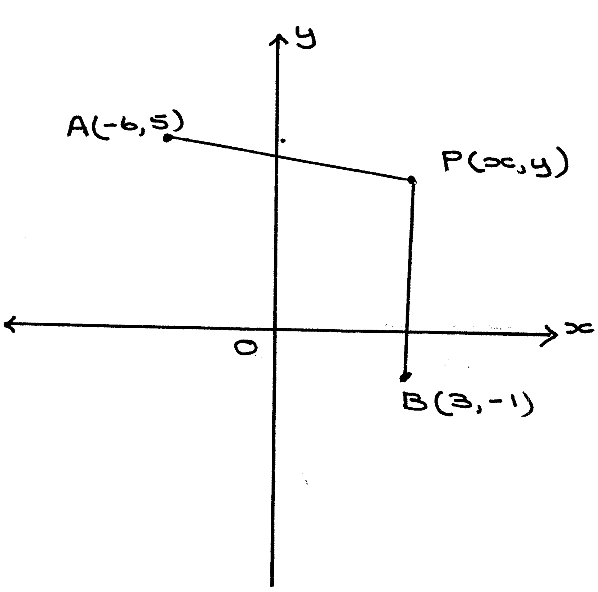
$\therefore$ directrix is $x = -11$

(ii) $(y-k)^2 = 4a(x-h)$

$(y+2)^2 = 4 \times 4(x+7)$

$(y+2)^2 = 16(x+7)$

e)



$PA = PB$

$\sqrt{(x+6)^2 + (y-5)^2} = \sqrt{(x-3)^2 + (y+1)^2}$

$\therefore PA^2 = PB^2$

$(x+6)^2 + (y-5)^2 = (x-3)^2 + (y+1)^2$

$x^2 + 12x + 36 + y^2 - 10y + 25 = x^2 - 6x + 9 + y^2 + 2y + 1$

$12x + 6x - 10y - 2y + 61 - 10 = 0$

$18x - 12y - 51 = 0$

$6x - 4y - 17 = 0$

Question 3

a) (i) $\int (4x^3 + 7x^2 - 3)\,dx = x^4 + \frac{7}{3}x^3 - 3x + C$

(ii) $\int \frac{6x^3 - 7x}{x^2}\,dx = \int \left(6x - \frac{7}{x}\right)dx$

$= 3x^2 - 7\log_e x + C$

(iii) $\int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx$

$= \frac{2}{5}x^{\frac{5}{2}} + C$

$= \frac{2}{5}x^2\sqrt{x} + C$

b) $\int_0^2 (2x-1)^3\,dx = \left[\frac{(2x-1)^4}{8}\right]_0^2$

$= \left[\frac{(2\times2-1)^4}{8}\right] - \left[\frac{(2\times0-1)^4}{8}\right]$

$= \frac{81}{8} - \frac{1}{8}$

$= 10$

c) $\frac{dy}{dx} = 3 + 2x - x^2$

$y = \int (3 + 2x - x^2)\,dx$

$= 3x + x^2 - \frac{1}{3}x^3 + C$

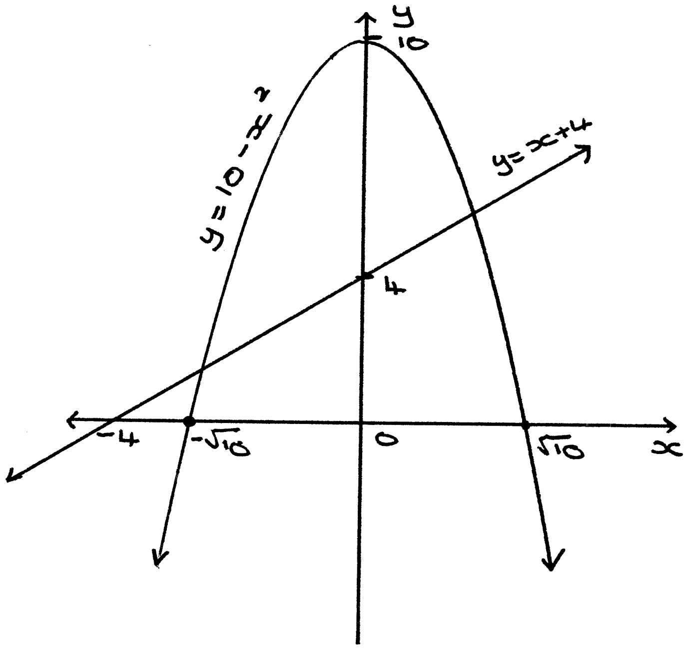
The curve passes through (2,5)

$\therefore 5 = 3(2) + 2^2 - \frac{1}{3}(2)^3 + C$

$5 = 6 + 4 - \frac{8}{3} + C$

$C = -\frac{7}{3}$

$\therefore y = -\frac{1}{3}x^3 + x^2 + 3x - \frac{7}{3}$



$$y = 10 - x^2 \quad \text{———} \quad ①$$

$$y = x + 4 \quad \text{———} \quad ②$$

① = ②

$$x + 4 = 10 - x^2$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$x = -3, 2$$

when $x = -3$

sub ②

$$y = x + 4$$
$$= -3 + 4$$
$$= 1$$

when $x = 2$

sub ②

$$y = x + 4$$
$$= 2 + 4$$
$$= 6$$

∴ points of intersection are $(-3, 1)$ $(2, 6)$

$$A = \int_{-3}^{2} (10 - x^2)\,dx - \int_{-3}^{2} (x+4)\,dx$$

$$= \int_{-3}^{2} (10 - x^2 - x - 4)\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$A = \left[6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-3}^{2}$$

$$A = \left[6(2) - \tfrac{1}{2}(2)^2 - \tfrac{1}{3}(2)^3\right] - \left[6(-3) - \tfrac{1}{2}(-3)^2 - \tfrac{1}{3}(-3)^3\right]$$

$$= \left[12 - 2 - \tfrac{8}{3}\right] - \left[-18 - \tfrac{9}{2} + 9\right]$$

$$= \frac{22}{3} + \frac{27}{2}$$

$$= 20\tfrac{5}{6}$$

∴ Area is $20\tfrac{5}{6}$ units^2

Question 4

a) (i) Let $y = e^{4x+8}$

$$\frac{dy}{dx} = 4e^{4x+8}$$

(ii) Let $y = \dfrac{e^x}{e^x+1}$

$$= \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad u = e^x \quad v = e^x + 1 \qquad \frac{du}{dx} = e^x \quad \frac{dv}{dx} = e^x$$

$$= \frac{e^x(e^x+1) - e^x . e^x}{(e^x+1)^2}$$

$$= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2}$$

$$= \frac{e^x}{(e^x+1)^2}$$

b) $$\int_0^2 e^{3x}\,dx = \left[\tfrac{1}{3}e^{3x}\right]_0^2$$

$$= \left[\tfrac{1}{3}e^{3(2)}\right] - \left[\tfrac{1}{3}e^{3(0)}\right]$$

$$= \tfrac{1}{3}e^6 - \tfrac{1}{3}$$

$$= \tfrac{1}{3}(e^6 - 1)$$

c) $\int 8x^3 e^{x^4}\,dx = 2\int 4x^3 e^{x^4}\,dx$

$= 2e^{x^4} + c$

d) $y = e^{x^2-1}$

$\frac{dy}{dx} = 2x\left(e^{x^2-1}\right)$ at $x = 1$

$= 2\left(e^{1^2-1}\right)$

$= 2$

when $x = 1$, $y = e^{x^2-1}$

$= e^{1-1}$

$= 1$

$\therefore$ point $(1,1)$ $\quad \frac{dy}{dx} = 2$

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x-1)$

$\therefore y = 2x - 1$

e) $y = e^{x} - e^{-x}$

$y^2 = (e^x - e^{-x})^2$

$= e^{2x} - 2 + e^{-2x}$

$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\frac{1}{2}} (e^{2x} - 2 + e^{-2x})\,dx$

$= \pi\left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_0^{\frac{1}{2}}$

$= \pi\left[\frac{1}{2}e^{2(\frac{1}{2})} - 2(\tfrac{1}{2}) - \frac{1}{2}e^{-2(\frac{1}{2})}\right] - \pi\left[\frac{1}{2}e^0 - 0 - \frac{1}{2}e^0\right]$

$= \pi\left[\frac{1}{2}e - 1 - \frac{1}{2}e^{-1}\right]$

$= \frac{\pi}{2}\left[e - 2 - \frac{1}{e}\right]$

$\therefore$ Volume is $\frac{\pi}{2}(e - 2 - \frac{1}{e})$ units^3

x	0	1	2	3	4
$f(x) = e^{-x^2}$	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}

$\int_a^b f(x)\,dx \doteq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$\int_0^4 e^{-x^2}\,dx \doteq \frac{2-0}{6}\left[1 + 4(e^{-1} + e^{-9}) + 2e^{-4} + e^{-16}\right]$

$\doteq 0.8362142647$

$= 0.84$ to 2dp.

Questions

a) (i) $y = \log_e(2x+7)$

$\frac{dy}{dx} = \frac{2}{2x+7}$

(ii) $y = \log_e(2x+1)(x-5)$

$= \log_e(2x^2 - 9x - 5)$

$\frac{dy}{dx} = \frac{4x-9}{2x^2-9x-5}$

(iii) $y = x^3 \log_e x$

$= uv$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= \log_e x \times 3x^2 + x^3 \times \frac{1}{x}$

$= 3x^2\log_e x + x^2$

$= x^2(3\log_e x + 1)$

b) $\int_2^3 \frac{8x}{2x^2+7}\,dx = 2\int_2^3 \frac{4x}{2x^2+7}\,dx$

$= \left[2\log_e(2x^2+7)\right]_2^3$

$= 2\log_e(2\times3^2+7) - 2\log_e(2\times2^2+7)$

$= 2\log_e 25 - 2\log_e 15$

$= 2\log_e\left(\frac{25}{15}\right)$

$= 2\log_e\left(\frac{5}{3}\right) = \log_e\left(\frac{25}{9}\right)$

c) i) $\frac{4x+3}{2x+1} = 2 + \frac{1}{2x+1}$

$L.HS = 2 + \frac{1}{2x+1}$

$= \frac{2(2x+1) \quad +1}{2x+1}$

$= \frac{4x+2+1}{2x+1}$

$= \frac{4x+3}{2x+1}$

$= R.HS$

$\therefore \frac{4x+3}{2x+1} = 2 + \frac{1}{2x+1}$

ii) $\int \frac{4x+3}{2x+1}\,dx = \int 2 + \frac{1}{2x+1}\,dx$

$= \int 2dx + \frac{1}{2}\int \frac{2x}{2x+1}dx$

$= 2x + \frac{1}{2}\log_e(2x+1) + c$

d) $2\log_3 6 + \log_3 18 - 3\log_3 2$

$= \log_3 6^2 + \log_3 18 - \log_3 2^3$

$= \log_3\left(\frac{36\times 18}{8}\right)$

$= \log_3 81$

$= \log_3 3^4$

$= 4\log_3 3$

$= 4$

e) (i) $\log_2 64 = x$

$2^x = 64$

$2^x = 2^6$

$x = 6$

(ii) $3^x = 5$

$\log_e 3^x = \log_e 5$

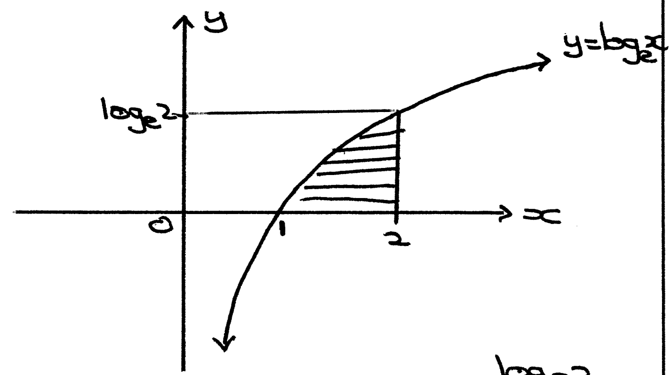
$x\log_e 3 = \log_e 5$

$x = \frac{\log_e 5}{\log_e 3}$

$\doteqdot 1.464973521$

$= 1.46$ to 2dp

f)



$A = \text{Area rectangle} - \int_0^{\log_e 2} e^y\,dy$

$= 2\times\log_e 2 - \left\{\left[e^{\log_e 2} - e^0\right]\right\}$

$= 2\log_e 2 - \{2-1\}$

$= 2\log_e 2 - 1$

$\therefore$ Area is $(2\log_e 2 - 1)$ units^2